Lemma) (contraction mapping)

Let B be a Banach space and $T: B \to B$.

suppose $\exists\, \theta < 1$ s.t.

$$\|Tx - Ty\| \leq \theta \|x - y\|.$$

Then, $\exists$ a unique $x \in B$ s.t.

$$Tx = x.$$

proof) suppose $Tx = x$, $Ty = y$

$$\Rightarrow \|x-y\| = \|Tx - Ty\| \leq \theta \|x-y\|$$

$$\Rightarrow \|x-y\| = 0, \quad x = y. \quad \text{unique!!}$$

Choose a $x_0 \in B$.

Define $x_1 = Tx_0$, $x_2 = Tx_1$, $x_3 = Tx_2$ - -

$$\|x_2 - x_1\| = \|Tx_1 - Tx_0\| \leq \theta \|x_1 - x_0\|$$

$$\|x_3 - x_2\| \leq \theta \|x_2 - x_1\| \leq \theta^2 \|x_1 - x_0\|$$

$\Rightarrow \|x_{m+k} - x_m\|$

$\leq \|x_{m+k} - x_{m+k-1}\| + \dots + \|x_{m+1} - x_m\|$

$\leq \theta^{m+k-1} \|x_1 - x_0\| + \dots + \theta^m \|x_1 - x_0\|$

$= (\theta^{k-1} + \dots + 1)\, \theta^m \|x_1 - x_0\|$

$\leq \dfrac{\theta^m}{1-\theta} \|x_1 - x_0\|$

$\Rightarrow \lim\limits_{m \to \infty} \|x_{m+k} - x_m\| = 0$

i.e. $\{x_m\}$ is a cauchy seq.

$\Rightarrow \exists\, \bar{x} = \lim\limits_{m \to \infty} x_m \in B$.

$T\bar{x} = \bar{x}$.

$\hookrightarrow \bar{x}$ is a fixed point.

Thm) (John Nash)

(Method of Continuity)

Let B be a Banach space.

V a normed linear space.

Let L_0, L_1 be linear operators

from B to V and

$$\|L_0 x\|_V \le C_0 \|x\|_B, \quad \|L_1 x\|_V \le C_1 \|x\|_B$$

For $t \in [0,1]$, set

$$L_t = tL_0 + (1-t)L_1 \qquad \forall t \in [0,1]$$

suppose $\exists\, C_2$ s.t. $\|x\|_B \le C_2 \|L_t x\|_V$.

Then, L_1 maps B onto V

iff L_0 maps B onto V.

Remark) $B = \{u \in C^{2,\alpha}(\bar{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$

$V = \{f \in C^{\alpha}(\bar{\Omega})\}$

$\|\cdot\|_B = \|\cdot\|_{C^{2,\alpha}}$, $\|\cdot\|_V = \|\cdot\|_{C^\alpha}$.

By the last theorem in Lecture 4.

$\|u\|_B \leq C_2 \|L_t u\|_V$

where $C_2 = C_2(n, \alpha, \Lambda, \lambda, \Omega)$.

$L_0 = \Delta$, $L_1 = a_{ij}\partial_i\partial_j + b_i\partial_i + c$

$\|\Delta u\|_V \leq n\|u\|_B$, $\|L_1 u\|_V \leq C_1 \|u\|_B$,

$C_1 = C_1(\Lambda, n)$.

Given $f \in V$, $\exists u \in B$ s.t. $\Delta u = f$

By Kellogg's thm. (i.e. $L_0 u = f$)

$\Rightarrow L_0$ maps B onto V.

By the method of cont. L_1 maps B onto V.

Cor) $\exists\, u \in C^{2,\alpha}(\bar{\Omega})$ s.t.

$Lu = f$ in Ω. $u = 0$ on $\partial\Omega$

if $c \le 0$. $a_{ij} = a_{ji}$, $a_{ij}\xi_i\xi_j \ge \lambda|\xi|^2$

$a_{ij}, b_i, c, f \in C^\alpha(\bar{\Omega})$ — $(*)$

Cor2) $\exists\, u \in C^{2,\alpha}(\bar{\Omega})$ s.t

$Lu = f$ in Ω. $u = g$ on $\partial\Omega$

If, we have $(*)$, $g \in C^{2,\alpha}(\bar{\Omega})$

pf of cor2) consider $w = u - g \in C^{2,\alpha}$

$Lw = Lu - Lg = f - Lg \in C^\alpha$

$w = 0$ on $\partial\Omega$.

Cor1 implies Cor 2.

pf of method of cont.)

Supp L_s is onto for some $s \in [0,1]$

$\|x\|_B \le c_2 \|L_t x\|_V \Rightarrow L_s$ is 1-1.

So, $L_s^{-1} : V \to B$ exists.

(i.e. $L_s^{-1} L_s x = x$)

For $t \in [0,1]$ and $z \in V$.

$$L_t x = z \iff L_s x = z + (L_s - L_t) x$$
$$= z + (s-t) L_0 x - (s-t) L_1 x.$$

$$\iff x = L_s^{-1} z + (s-t) L_s^{-1} (L_0 - L_1) x.$$

Define $Tx = L_s^{-1} z + (s-t) L_s^{-1} (L_0 - L_1) x \in B.$

$$Tx - Ty = (s-t) L_s^{-1} (L_0 - L_1)(x-y)$$

$$\Rightarrow \|Tx - Ty\|_B = |s-t| \, \|L_s^{-1} (L_0 - L_1)(x-y)\|_B$$
$$\le |s-t| \, c_2 \, \|(L_0 - L_1)(x-y)\|_V$$

$\Rightarrow \|Tx - Ty\|$

$$\leq C_2|s-t|\left(\|L_0(x-y)\|_V + \|L_1(x-y)\|_V\right)$$

$$\leq C_2|s-t|\left(C_0\|x-y\|_V + C_1\|x-y\|_V\right)$$

$$= C_2(C_0+C_1)|s-t|\,\|x-y\|_V.$$

if $|s-t| \leq \frac{1}{2} C_2^{-1}(C_0+C_1)^{-1} = \delta.$

then $\exists\ x \in B$ s.t. $Tx = x$.

$\Rightarrow$ given $z \in V$, $\exists$ a unique $x \in B$.

s.t. $L_t x = z$.

$\therefore$ if $t \in [s-\delta, s+\delta] \cap [0,1]$

then L_t maps B onto V.

$\boxtimes$

Section 5. Wave equation.

$$u_{tt} = \Delta u$$

for $u : \Omega \times [0,T] \to \mathbb{R}$

(1D example)

string w/ elasticity

mass.

↳ graph of $u(x,t)$

u_t : speed at x.

u_{tt} : accelation

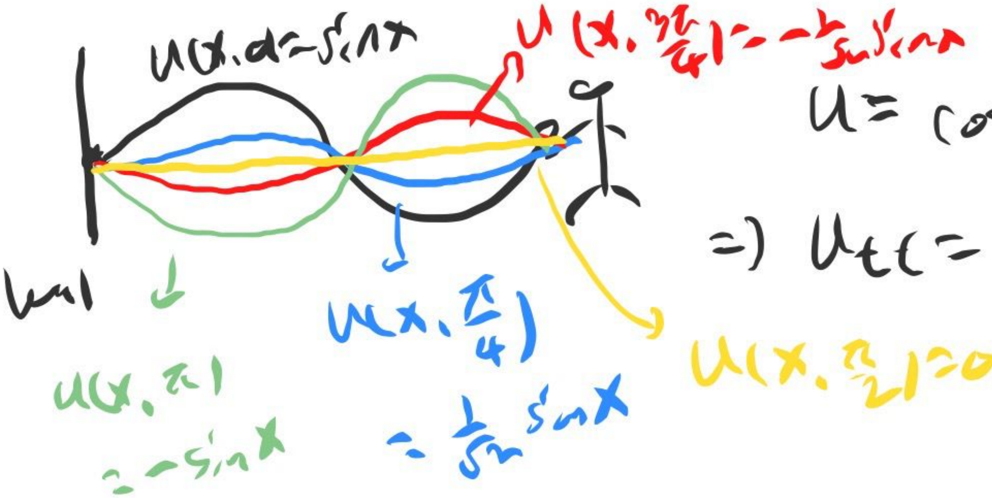
$F = ma = m u_{tt}$, $F = c u_{xx}$??

↑ force ↑ mass ↳ acc.

↳ $u_{tt} = \frac{c}{m} u_{xx}$

rubber band

Force



$u = \cos t \sin x$

$\Rightarrow u_{tt} = u_{xx}$

Ex) $n \geq 2$. $\Omega = \mathbb{R}^n$

$u(x,t) = \cos t \sin x$.

↳ sea wave

Thm) Energy Conservation.

Let $u_{tt} = \Delta u$ in Ω

$\partial_\nu u = 0$ on $\partial\Omega$.

Then. $E(t) = \int_\Omega |u_t|^2 + \|\nabla u\|^2 \, dx$

satisfies $E(t) = E(0)$.

Pf) $E' = 2\int_\Omega u_t u_{tt} + \nabla u \cdot \nabla u_t \, dx$

$$= 2\int_\Omega u_t u_{tt} + 2\int_{\partial\Omega} \cancel{u_\nu}^{\,0} u_t \, dS - 2\int_\Omega (\Delta u) u_t \, dx$$

$$= 2\int u_t (\cancel{u_{tt} - \Delta u}) \, dx$$

$$= 0.$$

Remark) $E = \int \underbrace{|u_t|^2}_{\text{kinetic energy}} + \underbrace{\|\nabla u\|^2}_{\text{potential energy}}$

$a_{ij}u_{ij} = f$, $u_t = a_{ij}u_{ij} + f$

Elliptic

parabolic

$u_{tt} = a_{ij}u_{ij} + f$

→ Hyperbolic

if $\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2$

$0 < \lambda \le \Lambda$

$u_{tt} = \partial_i(a_{ij}(x)u_j(x))$.

↳ divergence form.

Remark) wave eq does NOT satisfy the maximum principle.

Well-posedness

i.e. the conditions we need to assume for existence and uniqueness.

$\Delta u = f \Leftarrow$ Dirichlet or Neumann.

$\Delta u + f = u_t \Leftarrow$ Diri or Neu.

and Cauchy, $u(x,0) = g(x)$

$\Delta u + f = u_{tt} \Leftarrow$ Diri or Neu..

and Cauchy $u(x,0) = g(x)$
$u_t(x,0) = h(x)$

Ex) $u = \sin t \sin x$

$\Rightarrow u(x,0) = 0$

but $u_t(x,0) = \sin x \neq 0$.

Ex) $n=0$.

$u''(t) = 1$.

we need initial data $u(0), u'(0)$ for uniqueness.

1D separation of variables.

$I = (0, 2\pi)$. $u(0,t) = u(2\pi, t) = 0$

$u(x,0) = g$. $u_t(x,0) = h$.

$u_{tt} = u_{xx}$. $g, h, u \in C^\infty(I)$

Then., $u(x,t) = \sum_{m=1}^{\infty} a_m(t) \sin mx$

for each $t \in \mathbb{R}$.

$u_{xx} = -\sum m^2 a_m(t) \sin mx$

$\int_0^{2\pi} u_{xx} \sin mx \, dx = -m^2 a_m \int_0^{2\pi} (\sin mx)^2 dx$

$= -\pi m^2 a_m(t)$

$u_{tt} = \sum a_m''(t) \sin(mx)$

$\int_0^{2\pi} u_{tt} \sin mx \, dx = a_m''(t) \int_0^{2\pi} (\sin mx)^2 dx$

$= \pi a_m''(t)$.

$\therefore \; a_m'' = -m a_m$

$\Rightarrow \; a_m = C_1 \sin mt + C_2 \cos mt.$

$\Rightarrow a_m(0) = C_2 \; , \; a_m'(0) = m C_1$

On the other hand

$g(x) = u(x,0) = \sum a_m(0) \sin mx$

$$\int_0^{2\pi} g(x) \sin mx \, dx = a_m(0) \int_0^{2\pi} (\sin mx)^2$$

$$= \pi a_m(0).$$

$$\Rightarrow a_m(0) = \frac{1}{\pi} \int_0^{2\pi} g(x) \sin mx \, dx \triangleq \beta_m = C_2$$

$h(x) = u_t(x,0) = \sum a_m'(0) \sin mx$

$$\Rightarrow C_1 = \frac{1}{m} a_m'(0) = \underbrace{\frac{1}{m\pi} \int_0^{2\pi} h(x) \sin mx \, dx}_{\triangleq \alpha_m}.$$

$$\Rightarrow u(x,t) = \sum_{m=1}^{\infty} (\alpha_m \sin mt + \beta_m \cos mt) \sin mx.$$